

# Recursive Representation of Wronskians in Confluent Supersymmetric Quantum Mechanics

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## Abstract

A recursive form of arbitrary-order Wronskian associated with transformation functions in the confluent algorithm of supersymmetric quantum mechanics (SUSY) is constructed. With this recursive form regularity conditions for the generated potentials can be analyzed. Moreover, as byproducts we obtain new representations of solutions to Schrödinger equations that underwent a confluent SUSY-transformation.

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## 1 Introduction

The formalism of supersymmetry (SUSY) is one of the most significant methods for the construction of new solvable quantum models that feature a prescribed energy spectrum. Based on the mathematical concept of Darboux transformations that were first introduced in [11], the SUSY formalism interrelates quantum systems by means of linear differential operators (SUSY transformation). While such interrelated quantum systems are referred to as SUSY partners, the same terminology is commonly applied to their respective potentials. Since there is a vast amount of literature on the topic that encompasses many applications to particular quantum models, we refer the reader to the self-contained reviews [10] [14] [18] and references therein. The SUSY formalism can be split into two different cases that we call standard and confluent SUSY algorithm. The standard algorithm is relatively well understood and used in most of the applications that can be found in the literature. Its application requires to determine solutions to the governing equation of the initial quantum system at pairwise different energies, these solutions are called transformation functions. The transformed system is then characterized by the Wronskian of the transformation functions. A comprehensive and very detailed review of the standard SUSY algorithm and applications can be found in [3]. In contrast to the latter standard algorithm, its confluent counterpart is less known. The simplest version of SUSY known as first-order SUSY has the restriction that only spectral modifications can be done below the ground state without introducing new singularities in constructed potential, in order to make more general manipulations of the spectrum an iteration can be done [4]. If the same transformation function is cleverly used during the iteration an energy level can be introduced or deleted, when only one iteration is considered the equivalence between this technique and the Levitan-Gelfand procedure can be proven [15, 1, 5], this whole process is known as the confluent SUSY algorithm. This technique has been studied in particular contexts, for example the application of second

and third-order SUSY transformations to certain quantum systems [8] [13] or the construction of orthogonal polynomials through arbitrary-order SUSY transformations [16], just to name a few. In the confluent SUSY algorithm, the transformation functions form a Jordan chain of the Hamiltonian that governs the initial quantum problem, that is, they are generalized eigenvectors of that Hamiltonian [2]. As such, the transformation functions admit both an integral and a differential representation [20] [7], the relationship between which was reported on recently [9] [19]. One of the principal open questions regarding the confluent SUSY algorithm concerns the regularity of the potential in the SUSY-transformed system. In order to not feature singularities, the Wronskian of the associated transformation functions is not allowed to have zeros inside the domain of the system. While for the standard SUSY algorithm this can be established by means of regularity conditions [3], such conditions are only known for second and third-order confluent SUSY transformations [13], but not for higher orders. This is so because the regularity conditions emerge from the Wronskian of the transformation functions, the general form of which is known for second and third-order confluent SUSY transformations only. Therefore, the purpose of this work is to construct such a general form of the Wronskian for arbitrary-order confluent SUSY transformations. We approach this problem by constructing a recursive formula of the latter Wronskian for confluent SUSY transformations of arbitrary order. The advantages of this formula are twofold: first, we have a representation for the Wronskian which makes it much easier to derive regularity conditions for the SUSY-transformed potential. Second, since our formula allows to build a Wronskian from its lower-order counterparts, the calculation of transformation functions can be avoided. As a byproduct of our formula, we obtain alternative representations for SUSY-transformed solutions of the system's governing equations. In section 2 we summarize basic facts about the confluent SUSY algorithm. Section 3 is devoted to the construction of our recursive formula for the Wronskian associated with the transformation functions. In section 4 we present applications of our results in fourth and fifth-order confluent SUSY transformations.

## 2 Supersymmetric quantum mechanics

The SUSY formalism has its origin in the SUSY algebra, a graded Lie algebra of grade one. This algebra can be represented as the direct sum  $S = P \oplus L$ , where  $P$  stands for the Poincare algebra and the space  $L$  is spanned by two generators  $Q_1$  and  $Q_2$ :

$$Q_1 = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix} \quad Q_2 = -i \begin{pmatrix} 0 & A^\dagger \\ -A & 0 \end{pmatrix}. \quad (1)$$

Here,  $A$  and  $A^\dagger$  are adjoint differential operators of order  $n$  for two Schrödinger Hamiltonians  $H_0$  and  $H_1$ , that is,

$$H_1 A^\dagger = A^\dagger H_0. \quad (2)$$

The generators  $Q_1, Q_2$  satisfy the following anticommutator and commutator relations

$$\{Q_i, Q_j\} = \delta_{ij} H_S \quad [Q_i, H_S] = 0, \quad i, j = 1, 2,$$

where  $H_S$  stands for the operator

$$H_S = \begin{pmatrix} A^\dagger A & 0 \\ 0 & A A^\dagger \end{pmatrix}.$$

We can relate this operator to the pair of Hamiltonians  $H_0$  and  $H_1$  in (2) by means of the operator factorization

$$H_S = \prod_{j=0}^{n-1} \left[ \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix} - \lambda_j \right],$$

where  $\lambda_j$ ,  $j = 0, \dots, n-1$ , are complex-valued constants. Before we can comment further on the nature of these constants, we must distinguish the standard and the confluent SUSY algorithms. In the standard algorithm, the constants are pairwise different, that is, we have  $\lambda_i \neq \lambda_j$  for  $i, j = 0, \dots, n-1$ ,  $i \neq j$ . A value  $\lambda_j$  is associated with a solution  $u_j$  of the Schrödinger equation  $(H_0 - \lambda_j)u_j = 0$ . Observe that  $\lambda_j$  is not required to be in the spectrum of  $H_0$ . In the confluent algorithm we have  $\lambda_j = \lambda$ ,  $j = 0, \dots, n-1$ , that is, all constants are equal to each other. The constant  $\lambda$  is associated with  $n$  solutions  $u_j$ ,  $j = 0, \dots, n-1$ , of the equation  $(H_0 - \lambda)^n u_j = 0$ . If  $\lambda$  belongs to the discrete spectrum of  $H_0$ , then the functions  $u_j$  are called generalized eigenvectors or Jordan chain. Let us now assume that the operator  $H_0$  admits a discrete spectrum  $(E_j)$  and a family of associated eigenfunctions  $(\Psi_j) \subset L^2(D)$ , where  $D$  is the domain of  $V_0$ . Then, for a fixed  $j$  the function

$$\Phi_j = A^\dagger \Psi_j, \quad (3)$$

is an eigenfunction to the Hamiltonian  $H_1$  for the spectral value  $\lambda_j$ , provided  $A^\dagger \Psi_j \neq 0$ . The operator  $A^\dagger$  in (3) can be expressed through Wronskians, as will be demonstrated in the subsequent two paragraphs. The Hamiltonian  $H_1$  admits a complete set of eigenfunctions consisting of the  $\Phi_j$  defined in (3) and of solutions  $\Phi_{\lambda_j}$  of the Schrödinger equation  $(H_0 - \lambda_j)\Phi_{\lambda_j} = 0$ , provided  $\lambda_j$  is an eigenvalue of  $H_0$ . In the two subsequent paragraphs we will specify computational details regarding the transformation (3) and the potentials associated with our Hamiltonians  $H_0$ ,  $H_1$  for the standard and the confluent SUSY algorithm, respectively. Our starting point for reviewing the two SUSY algorithms is the one-dimensional stationary Schrödinger equation associated with the Hamiltonian  $H_0$ . We can write it in the form

$$\Psi'' + (E - V_0) \Psi = 0, \quad (4)$$

where the energy  $E$  is a real-valued constant and  $V_0$  denotes the potential.

## 2.1 The standard SUSY algorithm

For a natural number  $n$ , assume that  $u_0, u_1, \dots, u_{n-1}$  are solutions to equation (4), associated to the pairwise different energies  $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ , respectively. While the latter functions are often referred to as transformation functions or auxiliary solutions, their associated energies are usually called factorization energies. Now, the function

$$\Phi = \frac{W_{u_0, u_1, \dots, u_{n-1}, \Psi}}{W_{u_0, u_1, \dots, u_{n-1}}}, \quad (5)$$

where each  $W$  stands for the Wronskian of the functions in its index, is a solution to the transformed equation

$$\Phi'' + (E - V_n) \Phi = 0, \quad (6)$$

the potential  $V_n$  of which is related to its initial counterpart  $V_0$  as

$$V_n = V_0 - 2 \frac{d^2}{dx^2} \log (W_{u_0, u_1, \dots, u_{n-1}}). \quad (7)$$

Note that (5) corresponds to (3) and that  $V_n$  is the potential associated with the Hamiltonian  $H_1$ . The function (5) is called a SUSY transformation of order  $n$  ( $n$ -SUSY transformation), also called Darboux or Darboux-Crum transformation. The latter two names refer to the mathematical origin of the SUSY algorithms, see [11] and [3] for details. Note that the family  $(u_1, u_2, \dots, u_n, \Psi)$  must be linearly independent in order to avoid that (5) vanishes. If (4) is the governing equation of a spectral problem for the spectral parameter  $E$ , then a SUSY transformation of order  $n$  can change the spectrum of the transformed problem by at most  $n$  values, depending on the

factorization energies  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Therefore, SUSY transformations are used for generating potentials that are associated with a prescribed spectrum (spectral design). While any  $n$ -SUSY transformation can be performed as  $n$  successive first-order SUSY transformations, this increases the computational effort. Observe further that the transformation (5) does not necessarily yield a physically meaningful result unless the transformation functions are properly chosen. Particularly in the case of transforming bound-state solutions, properties like normalizability and orthogonality of the transformation outcomes are not guaranteed.

## 2.2 The confluent SUSY algorithm.

In order to apply a  $n$ -th order confluent SUSY transformation to (4), we first determine  $n + 1$  functions  $u_0, u_1, \dots, u_n$ , that solve the following system of equations,

$$u_0'' + (\lambda - V_0) u_0 = 0 \quad (8)$$

$$u_j'' + (\lambda - V_0) u_j = -u_{j-1}, \quad j = 1, \dots, n, \quad (9)$$

introducing a real constant  $\lambda$  that we assume to be different from  $E$  in (4). Solutions to the system (8), (9) is commonly referred to as Jordan chain of order  $n$ . In the standard SUSY scheme, the auxiliary solutions must satisfy the initial Schrödinger equation at pairwise different energies [3], which precisely constitutes the difference to the confluent algorithm that we are focusing on: here, all auxiliary solutions are associated with the same energy value  $\lambda$ . Now, once the system (8), (9) has been solved, we take a solution  $\Psi$  of our initial Schrödinger equation (4) and construct the following functions  $\Phi_n$  and  $\chi_n$ :

$$\Phi_n = \frac{W_{u_0, \dots, u_{n-1}, \Psi}}{W_{u_0, \dots, u_{n-1}}}, \quad \chi_n = \frac{W_{u_0, \dots, u_n}}{W_{u_0, \dots, u_{n-1}}}, \quad (10)$$

where the symbol  $W$  stands for the Wronskian of the functions in its index. Then,  $\Phi_n$  and  $\chi_n$  are solutions to the following Schrödinger equations

$$\Phi_n'' + (E - V_n) \Phi_n = 0 \quad \chi_n'' + (\lambda - V_n) \chi_n = 0, \quad (11)$$

recall that we required  $\lambda \neq E$ . The transformed potential  $V_n$  is given by the expression

$$V_n = V_0 - 2 \frac{d^2}{dx^2} \log (W_{u_0, u_1, \dots, u_{n-1}}). \quad (12)$$

As in case of the standard SUSY algorithm, confluent transformations deliver physically meaningful results only if the transformation functions are chosen appropriately. While the expressions for the transformed solutions (10) and its associated potential (12) look formally the same as in the conventional SUSY scheme, they are profoundly different due to the system (8), (9) that determines the transformation functions in the confluent case. These functions admit an integral and a differential representation. The first of these representations can be constructed by means of the variations-of-constants formula [20]:

$$u_j = \hat{u} - u_0 \int \left( \int_0^t u_0 u_{j-1} ds \right) \frac{1}{u_0^2} dt, \quad j = 1, \dots, n-1, \quad (13)$$

where  $\hat{u}$  stands for any solution of the first equation (8). An alternative representation for the transformation functions involves parametric derivatives with respect to  $E$  [7]. Assuming that any solution of (8) is a function of the two variables  $x$  and  $\lambda$ , we have

$$u_j = \sum_{k=0}^{j-1} \frac{\partial \hat{u}_k}{\partial \lambda^k} + \frac{1}{j!} \frac{\partial u_0}{\partial \lambda^j}, \quad j = 1, \dots, n-1, \quad (14)$$

where  $\hat{u}_k$ ,  $k = 0, \dots, n-2$ , stand for arbitrary solutions of (8), including the trivial zero solution. Note that the representations (13) and (14) are not equivalent, but related to each other [9] [19]. Similar to the standard case, confluent SUSY transformations can be used for spectral design, but allow for the change of a single spectral value only. As such, the principal purpose of higher-order confluent SUSY transformations is the generation of new potentials that render the associated Schrödinger equation exactly-solvable. A typical application can be found in [16].

### 3 Recursive representation of the Wronskian

For the most part SUSY transformations are used to generate systems that are physically meaningful. As an important aspect of this, the transformed potential should either remain free of singularities or at least not receive additional singularities by undergoing the SUSY transformation. The principal quantity in the SUSY algorithms that controls the potential's singularities is the Wronskian of the transformation functions. As can be seen from (7) and (12), any zero of the Wronskian contributes a singularity in the transformed potential. We are therefore interested in choosing the transformation function such that our Wronskian remains free of singularities. In case of second- and third-order confluent SUSY transformations this was achieved by means of constructing a closed-form expression of the Wronskian [13]. In the following we will generalize this construction to arbitrary-order confluent SUSY transformations with the purpose to derive regularity conditions for the potentials resulting from such transformations.

#### 3.1 Construction of a recursion formula

We will now derive a representation of our Wronskian through a particular case of the system (8), (9). More precisely, let us consider the following pair of equations

$$\chi_{n-1}'' + (\lambda - V_{n-1}) \chi_{n-1} = 0 \quad (15)$$

$$\xi'' + (\lambda - V_{n-1}) \xi = -\chi_{n-1}. \quad (16)$$

We observe that these two equations are counterparts of (8), (9) for  $n = 1$ . In contrast to the latter system, (15) and (16) apply to a Schrödinger equation that underwent a confluent SUSY transformation of order  $n - 1$ . This Schrödinger equation is shown on the right side of (11) if we replace  $n$  by  $n - 1$ . Let us now calculate the Wronskian of the functions  $\chi_{n-1}$  and  $\xi$ . Its derivative reads

$$W'_{\chi_{n-1}, \xi} = \chi_{n-1} \xi'' - \chi_{n-1}'' \xi.$$

We replace the second derivatives by means of equations (15), (16). This gives

$$\begin{aligned} W'_{\chi_{n-1}, \xi} &= \chi_{n-1} (V_{n-1} - \lambda) \xi - \chi_{n-1}^2 - (V_{n-1} - \lambda) \chi_{n-1} \xi \\ &= -\chi_{n-1}^2. \end{aligned}$$

Consequently, integration on both sides reveals our Wronskian in the form

$$W_{\chi_{n-1}, \xi} = - \int_{\lambda}^x \chi_{n-1}^2 dt. \quad (17)$$

This result is not surprising because it follows immediately from the known relation

$$W_{u_0, u_1} = - \int_{\lambda}^x u_0^2 dt. \quad (18)$$

Let us now keep (17) in mind, while we construct another form of the Wronskian  $W_{\chi_{n-1},\xi}$ . According to its definition, we have

$$W_{\chi_{n-1},\xi} = \chi_{n-1} \xi' - \chi'_{n-1} \xi = \chi_{n-1} \left( \xi' - \frac{\chi'_{n-1}}{\chi_{n-1}} \xi \right).$$

Now, the term in parenthesis can be interpreted as a SUSY transformation of first order, resulting in a solution  $\chi_n$  of the Schrödinger equation on the right side of (11). We have

$$W_{\chi_{n-1},\xi} = \chi_{n-1} \chi_n. \quad (19)$$

After combining the results (17) and (19), we find

$$\chi_{n-1} \chi_n = - \int^x \chi_{n-1}^2 dt. \quad (20)$$

Since the functions  $\chi_n$  and  $\chi_{n-1}$  obey the representation on the right side of (10), we can substitute the latter representation in (20):

$$\frac{W_{u_0,\dots,u_{n-1}}}{W_{u_0,\dots,u_{n-2}}} \frac{W_{u_0,\dots,u_n}}{W_{u_0,\dots,u_{n-1}}} = - \int^x \left( \frac{W_{u_0,\dots,u_{n-1}}}{W_{u_0,\dots,u_{n-2}}} \right)^2 dt.$$

After minor simplifications we obtain the following result

$$W_{u_0,\dots,u_n} = -W_{u_0,\dots,u_{n-2}} \int^x \left( \frac{W_{u_0,\dots,u_{n-1}}}{W_{u_0,\dots,u_{n-2}}} \right)^2 dt. \quad (21)$$

This is a recursive formula for the Wronskian of the transformation functions  $u_j$ ,  $j = 0, \dots, n$ , in the confluent SUSY algorithm. It is important to point out that the parameter  $n$  in (21) can take arbitrarily high, but finite values. As an example, let us evaluate (21) for the particular case  $n = 2$ . We obtain

$$W_{u_0,u_1,u_2} = -u_0 \int^x \left( \frac{W_{u_0,u_1}}{u_0} \right)^2 dt.$$

This coincides precisely with the result found in [13]. Besides providing a representation of the Wronskian on its left side, our identity (21) can be used to determine SUSY-transformed potentials (12) without the need to calculate the transformation functions  $u_j$ ,  $j \geq 1$ . We will illustrate this property in our application section 3.

### 3.2 Factorization properties

A closer inspection of our recursive formula (21) reveals that upon iterating it, the Wronskian on its left side takes a factorized form. We will now construct this form and determine its factors. As a first step, let us now use our formula (21) to replace the first factor  $W_{u_0,\dots,u_{n-2}}$  on its right side. We obtain

$$\begin{aligned} W_{u_0,\dots,u_n} &= - \left[ -W_{u_0,\dots,u_{n-4}} \int^x \left( \frac{W_{u_0,\dots,u_{n-3}}}{W_{u_0,\dots,u_{n-4}}} \right)^2 dt \right] \left[ \int^x \left( \frac{W_{u_0,\dots,u_{n-1}}}{W_{u_0,\dots,u_{n-2}}} \right)^2 dt \right] \\ &= W_{u_0,\dots,u_{n-4}} \left[ \int^x \left( \frac{W_{u_0,\dots,u_{n-1}}}{W_{u_0,\dots,u_{n-2}}} \right)^2 dt \right] \left[ \int^x \left( \frac{W_{u_0,\dots,u_{n-3}}}{W_{u_0,\dots,u_{n-4}}} \right)^2 dt \right]. \end{aligned}$$

If this procedure is iterated, we obtain the following representation of our Wronskian

$$W_{u_0, \dots, u_n} = \begin{cases} (-1)^{\frac{n}{2}} u_0 \prod_{\substack{j=1 \\ j \text{ odd}}}^{n-1} \left[ \int^x \left( \frac{W_{u_0, \dots, u_{n-j}}}{W_{u_0, \dots, u_{n-j-1}}} \right)^2 dt \right] & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} W_{u_0, u_1} \prod_{\substack{j=1 \\ j \text{ odd}}}^{n-2} \left[ \int^x \left( \frac{W_{u_0, \dots, u_{n-j}}}{W_{u_0, \dots, u_{n-j-1}}} \right)^2 dt \right] & \text{if } n \text{ is odd} \end{cases}.$$

While this is already the sought factorized form, we can say a bit more about the actual factors. To this end, we observe that the quotient inside the integrals can be matched with the functions on the right side of (10), such that we have

$$W_{u_0, \dots, u_n} = \begin{cases} (-1)^{\frac{n}{2}} u_0 \prod_{\substack{j=1 \\ j \text{ odd}}}^{n-1} \left[ \int^x \chi_{n-j}^2 dt \right] & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} W_{u_0, u_1} \prod_{\substack{j=1 \\ j \text{ odd}}}^{n-2} \left[ \int^x \chi_{n-j}^2 dt \right] & \text{if } n \text{ is odd} \end{cases}. \quad (22)$$

Recall that according to (11) and (12), a function  $\chi_{n-j}$  solves the Schrödinger equation

$$\chi_{n-j}'' + \left[ \lambda - V_0 + 2 \frac{d}{dx} \left( \frac{W'_{u_0, \dots, u_{n-j-1}}}{W_{u_0, \dots, u_{n-j-1}}} \right) - C \right] \chi_{n-j} = 0,$$

where  $j$  runs through odd natural numbers up to  $n-1$  if  $n$  is even or  $n-2$  if  $n$  is odd. In the final step we apply the recursive relation (20) between the functions  $\chi_j$ ,  $j = 1, \dots, n-1$  to (22), which is reduced to the simple form

$$W_{u_0, \dots, u_n} = \prod_{j=0}^n \chi_j, \quad (23)$$

where we defined  $\chi_0 = u_0$ . Since for a fixed  $j$  the function  $\chi_j$  can be written as a first-order SUSY transformation of  $\chi_{j-1}$ , the representation (23) provides a factorization of our Wronskian in terms of SUSY transformation (or Darboux) operators. Such a factorization is known to exist for the standard SUSY formalism [3]. Note that this factorization as well as (23), are not to be confused with the factorization due to Infeld and Hull [17], as the latter method does not apply to Wronskians, but to Hamiltonians.

### 3.3 Representations of SUSY transformations

In this section we will show that the recursive form of the Wronskian (21) leads to representations of SUSY transformations that can be used as an alternative to (10). The first of these representations is concerned with the solution  $\chi_n$  of the Schrödinger equation on the right side of (11). The latter equation admits its general solution to be written as a linear combination of two linearly independent functions, one of which is  $\chi_n$  and the second function we call  $\chi_n^\perp$ . We can find  $\chi_n^\perp$  through the reduction of order formula. This formula states that the solutions  $\chi_n$  and  $\chi_n^\perp$  are related as

$$\chi_n = \chi_n^\perp \int^x \frac{1}{(\chi_n^\perp)^2} dt, \quad (24)$$

their Wronskian being given by  $W_{\chi_n^\perp, \chi_n} = 1$ . Keeping this in mind, we rewrite the function  $\chi_n$  in (10) using our recursive formula (21). This gives

$$\chi_n = \frac{W_{u_0, \dots, u_{n-2}}}{W_{u_0, \dots, u_{n-1}}} \int^x \left( \frac{W_{u_0, \dots, u_{n-1}}}{W_{u_0, \dots, u_{n-2}}} \right)^2 dt. \quad (25)$$

Now, comparison between (24) and (25) leads to the conclusion that  $\chi_n^\perp$  has the form

$$\chi_n^\perp = \frac{W_{u_0, \dots, u_{n-2}}}{W_{u_0, \dots, u_{n-1}}}. \quad (26)$$

Hence, the general solution of the Schrödinger equation on the right of (11) can be expressed in terms of Wronskians. As a second application of our formula (21) we will now derive an alternative expression for the function  $\Phi_n$  in (10). Before we start, let us point out that the definition of  $\Phi_n$  in (10) depends on the Wronskian  $W_{u_0, \dots, u_{n-1}, \Psi}$ . This will play a role below. Now, in order to obtain an alternative expression to (10) for  $\Phi_n$ , a procedure similar to the one for obtaining (21) can be followed. Let us consider the set of equations

$$\chi_{n-1}'' + (\lambda - V_{n-1}) \chi_{n-1} = 0 \quad (27)$$

$$\Phi_{n-1}'' + (E - V_{n-1}) \Phi_{n-1} = 0. \quad (28)$$

The derivative of the Wronskian  $W_{\chi_{n-1}, \Phi_{n-1}}$  obeys the form

$$W_{\chi_{n-1}, \Phi_{n-1}}' = \chi_{n-1} \Phi_{n-1}'' - \chi_{n-1}' \Phi_{n-1} = (\lambda - E) \chi_{n-1} \Phi_{n-1}. \quad (29)$$

After integrating both sides of the previous equation we obtain

$$W_{\chi_{n-1}, \Phi_{n-1}} = (\lambda - E) \int^x \chi_{n-1} \Phi_{n-1} dt. \quad (30)$$

Alternatively, using the definition of a Wronskian, we get

$$W_{\chi_{n-1}, \Phi_{n-1}} = \chi_{n-1} \Phi_{n-1}' - \chi_{n-1}' \Phi_{n-1} = \chi_{n-1} \left( \Phi_{n-1}' - \frac{\chi_{n-1}'}{\chi_{n-1}} \Phi_{n-1} \right) = \chi_{n-1} \Phi_n. \quad (31)$$

The left and right side of this identity can be combined to yield

$$\Phi_n = \frac{W_{\chi_{n-1}, \Phi_{n-1}}}{\chi_{n-1}}.$$

In the final step we replace numerator and denominator on the right side by (30) and (10), respectively. This gives

$$\Phi_n = (\lambda - E) \frac{W_{u_0, \dots, u_{n-2}}}{W_{u_0, \dots, u_{n-1}}} \int^x \left( \frac{W_{u_0, \dots, u_{n-1}}}{W_{u_0, \dots, u_{n-2}}} \right) \left( \frac{W_{u_0, \dots, u_{n-2}, \Psi}}{W_{u_0, \dots, u_{n-2}}} \right) dt. \quad (32)$$

Functions in (26) and (32) are solutions of the Schrödinger equation (11), and these representations are important when introducing integration constants in (21) which will generate different SUSY partners potentials  $V_n$ . Observe that the solution index  $n$  shown in (32) can take arbitrarily high values, for example when enumerating an infinite set of bound state solutions. As such, there is usually no limit of the sequence  $(\Phi_n)$  in  $L^2(D)$ , where  $D$  stands for the system's domain.



## 4 Applications

Let us now use our main results (21), (26) and (32) in an example. We consider the Schrödinger equation

$$\Psi'' + \left( E + \frac{2}{\cosh^2(x)} \right) \Psi = 0, \quad (33)$$

defined on the whole real line and equipped with boundary conditions  $\lim_{|x| \rightarrow \infty} \Psi(x) = 0$ . We observe that (33) is a particular case of (4) if the potential  $V_0$  is chosen as

$$V_0(x) = -\frac{2}{\cosh^2(x)}. \quad (34)$$

This interaction is known as the Pöschl-Teller potential [21]. Even though there is already a vast amount of literature on this potential and its SUSY partners, it serves as a good toy model for our purposes, since the SUSY partners and their associated solutions of the Schrödinger equation are short enough to be stated in full form. The spectral problem governed by equation (33) and its boundary conditions admits a discrete spectrum that contains the single value  $E = -1$ . This eigenvalue is associated with a bound state-solution  $\Psi \in L^2(\mathbb{R})$ , given by

$$\Psi = \frac{1}{\sqrt{2} \cosh(x)}. \quad (35)$$

Note that the numerical factor is included to ensure correct normalization. In order to perform higher-order SUSY transformations we need the transformation functions  $u_0$  and  $u_j$ ,  $j = 1, 2, \dots$ , that are solutions of (8) and (9), respectively. Once we have selected the function  $u_0$  we can construct the functions  $u_j$ ,  $j = 1, 2, \dots$ , through their integral representation (13). For the present example we use

$$u_0 = \sqrt{2\kappa} \exp(\kappa x) [\tanh(x) - \kappa], \quad (36)$$

where  $\kappa = \sqrt{-\lambda}$ , recall that  $\lambda$  is the factorization energy from (8), (9). Throughout this example we will assume that  $\lambda < 0$ , such that  $\kappa > 0$ . Notice that the solution  $u_0$  of (8) vanishes for  $x \rightarrow -\infty$  if  $\kappa > 0$ , this will be important for the regularity of the new potentials generated by our SUSY transformation. To perform fourth and fifth-order transformations the needed remaining transformation functions  $u_1, u_2, u_3$ , obtained using (13), are:

$$u_1 = -\frac{\exp(\kappa x)}{\sqrt{2\kappa^3}} [\kappa^2 x - (\kappa x - 1) \tanh(x)] \quad (37)$$

$$u_2 = \frac{\exp(\kappa x)}{4\sqrt{2\kappa^7}} \{ \kappa^2 x(1 - \kappa x) + [\kappa x(\kappa x - 3) + 3] \tanh(x) \} \quad (38)$$

$$u_3 = \frac{\exp(\kappa x)}{24\sqrt{2\kappa^{11}}} \{ -\kappa^2 x [\kappa x(\kappa x - 3) + 3] + [\kappa x(\kappa^2 x^2 - 6\kappa x + 15) - 15] \tanh(x) \} \quad (39)$$

Observe that in principle we need another transformation function for the fifth-order case. However, we will show that using our representations (21) and (32), we can avoid the computation of  $u_4$ . Let us point out that in principle we do not need to determine any transformation function except for  $u_0$ . Instead, we can iterate our formula (21) to successively generate the Wronskians  $W_{u_0, u_1}$ ,  $W_{u_0, u_1, u_2}$  and  $W_{u_0, u_1, u_2, u_3}$ . Since these three steps involve a large amount of calculations, we restrict ourselves here to showing the last step only in the next section.

#### 4.1 Fourth-order SUSY transformation

To perform a fourth-order SUSY transformation to our equation (33), we will use the functions  $u_0$  and  $u_1, u_2, u_3$  that are given in (36) and (37)-(39), respectively. Before applying the actual transformation, we need to make sure that the resulting transformed potential  $V_1$  in (11) is free of singularities on the real line. Since this is guaranteed if the Wronskian of the transformation functions does not vanish, we must analyze this Wronskian. Its explicit form can be obtained in two different ways, either by direct calculation using (36)-(39) or through our recursive formula (21). Using the latter formula, we obtain

$$W_{u_0, u_1, u_2, u_3} = -W_{u_0, u_1} \cdot \int^x \left( \frac{W_{u_0, u_1, u_2}}{W_{u_0, u_1}} \right)^2 dt. \quad (40)$$

Observe that the right side of this identity does not contain the transformation function  $u_3$ . Since the integration is indefinite, it involves an arbitrary additive constant that we call  $C_a$ . We can therefore rewrite (40) in a somewhat intuitive matter as

$$W_{u_0, u_1, u_2, u_3} = -W_{u_0, u_1} \cdot \left[ C_a + \int^x \left( \frac{W_{u_0, u_1, u_2}}{W_{u_0, u_1}} \right)^2 dt \right] \quad (41)$$

$$= \exp(2\kappa x) [1 + \kappa^2 - 2\kappa \tanh(x)] \times \left[ C_a + \exp(2\kappa x) \frac{(\kappa^4 + 6\kappa^2 + 1) \cosh(x) - 4(\kappa^3 + \kappa) \sinh(x)}{16\kappa^4 [(\kappa^2 + 1) \cosh(x) - 2\kappa \sinh(x)]} \right]. \quad (42)$$

Since in this relatively simple example we were able to calculate the transformation functions (36)-(39), let us verify the result (42) by calculating the Wronskian  $W_{u_0, u_1, u_2, u_3}$  directly. We obtain

$$W_{u_0, u_1, u_2, u_3} = \frac{1}{16\kappa^4} \exp(4\kappa x) [1 + 6\kappa^2 + \kappa^4 - 4(\kappa + \kappa^3) \tanh(x)]. \quad (43)$$

One verifies by direct calculation that this is a special case of (42), obtained by substituting  $C_a = 0$  and simplifying. It is clear that our result (43) cannot contain any free constants because all transformation functions are determined. Since in (42) we did not use our function  $u_3$ , the resulting expression for the Wronskian is determined up to an integration constant, which allows adjustment to yield a regular potential. Recall that we need this Wronskian to be free of zeros in order to avoid singularities in the SUSY-transformed potential. A further analysis shows that the integral in (41) is a monotonically increasing function that tends to zero when  $x$  goes to  $-\infty$ . As a direct consequence, if the condition  $C_a > 0$  is fulfilled, then the term in brackets on the right side of (42) does not have any zeros. The remaining factor  $W_{u_0, u_1}$  in the same expression can be shown to not contribute any zeros either, which is a direct consequence of  $u_0$  vanishing for  $x \rightarrow -\infty$ . We can now construct SUSY partner potentials  $V_4$  using (12) with  $n = 4$ ,  $V_0$  as in (34) and  $W_{u_0, u_1, u_2, u_3}$  as in (41), note that every nonnegative value of the integration constant  $C_a$  will lead to a different regular SUSY-transformed potential. The explicit expression of the family of partner potentials  $V_4$  is not displayed here because of its length, but since all functions involved are shown it can be seen that the final expression will involve only exponential and hyperbolic functions. A particular case of a potential  $V_4$  can be found in figure 1. Next we determine the SUSY-transformed counterpart of (35) by means of (10) for  $n = 4$  by substituting (35) and (42) into the representation (32). We obtain an eigenfunction  $\Phi_4 \in L^2(\mathbb{R})$ , associated with the eigenvalue  $E = -1$ , the explicit form of which reads

$$\Phi_4 = \frac{(\kappa^2 - 1)^2 \operatorname{sech}(x) [16C_a \kappa^4 - (\kappa^2 - 1) \exp(2\kappa x)]}{\sqrt{2} \{16C_a \kappa^4 [\kappa^2 - 2\kappa \tanh(x) + 1] + \exp(2\kappa x) [\kappa^4 + 6\kappa^2 - 4(\kappa^3 + \kappa) \tanh(x) + 1]\}}. \quad (44)$$

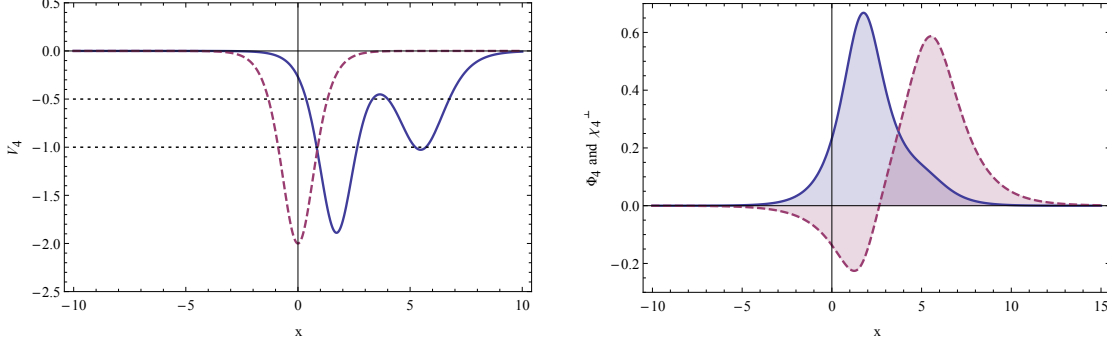


Figure 1: Left: Fourth-order SUSY partner of the hyperbolic Pöschl-Teller potential (blue solid curve). The Pöschl-Teller potential is plotted as reference (purple dashed curve), the parameters used are  $C_a = 50$ ,  $\kappa = 1/\sqrt{2}$ . Right: Eigenfunctions  $\Phi_4$  and  $\chi_4^\perp$  corresponding to the potential on the left.

The second eigenfunction of this 2-level system is  $\chi_4^\perp$  given by (26) with  $\lambda$  as corresponding eigenvalue and it is given by

$$\chi_4^\perp = \frac{\exp(\kappa x) [\kappa (\kappa^2 + 3) - (3\kappa^2 + 1) \tanh(x)]}{2\sqrt{2\kappa^3} [\kappa^2 - 2\kappa \tanh(x) + 1] \left\{ C_a + \frac{\exp(2\kappa x) [(\kappa^4 + 6\kappa^2 + 1) \cosh(x) - 4(\kappa^3 + \kappa) \sinh(x)]}{16\kappa^4 [(\kappa^2 + 1) \cosh(x) - 2\kappa \sinh(x)]} \right\}}. \quad (45)$$

In contrast to the initial problem that admits a single eigenvalue, the discrete spectrum of the transformed problem associated with the potential  $V_4$ , contains two eigenvalues. More precisely, the discrete spectrum is given by  $\{-1, \lambda\}$ , i. e., the SUSY transformation added the eigenvalue  $\lambda$  to the discrete spectrum. It is worth noticing that for any even transformation  $\lambda$  can be greater or less than the ground state energy  $E = -1$ . The left part of figure 1 shows a SUSY partner  $V_4$  (blue solid curve) with the parameters  $C_a = 50$ ,  $\kappa = 1/\sqrt{2}$  and the Pöschl-Teller potential  $V_0$  (see (34)) plotted as a reference (purple dashed curve), horizontal lines corresponding to  $E = -1$  and  $\lambda = -1/2$  are also plotted (black dotted curves). On the right of the figure, the only two eigenfunctions (normalized) of the corresponding Schrödinger equation for  $V_4$  are plotted,  $\Phi_4$  with eigenvalue  $E = -1$  (blue continuous curve) and the first excited state  $\chi_n^\perp$  (purple dashed curve) with  $\lambda = -1/2$  as corresponding eigenvalue.

## 4.2 Fifth-order SUSY transformation

For the construction of a SUSY partner potential to (34) using a fifth-order transformation, we proceed as in the previous example. As a first step, we compute the Wronskian  $W_{u_0, u_1, u_2, u_3, u_4}$  of the transformation functions and derive a condition for it to be free of zeros. We cannot find this Wronskian directly because we do not have the transformation function  $u_4$ . Instead, we will use once more our recursive representation (21) that only requires knowledge of the transformation functions  $u_0, u_1, u_2$  and  $u_3$ . Upon substituting (36)-(39) and including a constant of integration  $C_b$ , we arrive at

$$W_{u_0, u_1, u_2, u_3, u_4} = -W_{u_0, u_1, u_2} \cdot \left[ C_b + \int^x \left( \frac{W_{u_0, u_1, u_2, u_3}}{W_{u_0, u_1, u_2}} \right)^2 dt \right] \quad (46)$$

$$= \frac{\exp(3\kappa x)}{2\sqrt{2\kappa^3}} [-\kappa (\kappa^2 + 3) + (3\kappa^2 + 1) \tanh(x)] \\ \times \left\{ C_b + \exp(2\kappa x) \frac{\kappa (\kappa^4 + 10\kappa^2 + 5) \cosh(x) - (5\kappa^4 + 10\kappa^2 + 1) \sinh(x)}{64\kappa^6 [\kappa (\kappa^2 + 3) \cosh(x) - (3\kappa^2 + 1) \sinh(x)]} \right\}. \quad (47)$$

Using an argument similar to the fourth-order case, we can ensure that this Wronskian will not vanish on the real axis by imposing the constraint  $C_b \geq 0$ . Furthermore, when the Wronskian  $W_{u_0, u_1, u_2}$  on the right side of (46) is factored using again (21), it can be observed that  $u_0$  is a factor, thus  $u_0$  has to be chosen without any zeros. According to the Sturm oscillatory theorem [6] a necessary condition to have a non vanishing function  $u_0$  is that  $\lambda$  must be less than or equal to the ground state energy  $E = -1$ . Using this setting, a fifth-order SUSY potential  $V_5$  that is free of singularities can then be generated using (12), (34) and (47), note that an explicit expression of  $V_5$  is not displayed due to its length. A particular case of  $V_5$  is shown in figure 2. Let us point out that for the construction of this potential we did not have to compute the transformation function  $u_4$  by using the recursive formula (21). It remains to determine the solutions associated with the transformed potential  $V_5$ . To this end, we first observe that due to the constraint  $\lambda \leq -1$ , guaranteeing regularity of  $V_5$ , the solution associated with the lowest eigenvalue is given by  $\chi_5^\perp$ . This function can be constructed using (25) with  $W_{u_0, u_1, u_2, u_3, u_4}$  as in (47), its explicit form is then

$$\begin{aligned} \chi_5^\perp = & - \left\{ 8\sqrt{2\kappa^7} \exp(\kappa x) \left[ (\kappa^4 + 6\kappa^2 + 1) \cosh(x) - 4\kappa (\kappa^2 + 1) \sinh(x) \right] \right\} \times \\ & \times \left\{ 64C_b\kappa^6 \left[ \kappa (\kappa^2 + 3) \cosh(x) - (3\kappa^2 + 1) \sinh(x) \right] + \exp(2\kappa x) \times \right. \\ & \times \left. \left[ (\kappa^5 + 10\kappa^3 + 5\kappa) \cosh(x) - (5\kappa^4 + 10\kappa^2 + 1) \sinh(x) \right] \right\}^{-1}. \end{aligned}$$

The remaining solution  $\Phi_5$ , representing the first excited state of the system, can be obtained by evaluating (32). After some elementary simplification we arrive at

$$\Phi_5 = \frac{(\kappa^2 - 1)^3 \operatorname{sech}(x) [(\kappa^2 - 1) \exp(2\kappa x) - 64C_b\kappa^6]}{\sqrt{2} \{ 64C_b\kappa^6 [\kappa^3 + 3\kappa - (3\kappa^2 + 1) \tanh(x)] + \exp(2\kappa x) [\kappa^5 + 10\kappa^3 + 5\kappa - (5\kappa^4 + 10\kappa^2 + 1) \tanh(x)] \}}.$$

The discrete spectrum of the transformed problem for the potential  $V_5$  is  $\{\lambda, -1\}$ , that is, it contains two eigenvalues, one of which was created through the SUSY transformation. In the left part of figure 2 we see a fifth-order SUSY partner  $V_5$  (blue solid curve), generated with the parameters  $C_b = 0.01$  and  $\lambda = -3/2$ , the Pöschl-Teller potential is also displayed (purple dashed curve) and horizontal lines for the values  $E = -1$  and  $\lambda = -3/2$  are also plotted (black dotted lines). On the right of the same figure, the eigenfunctions  $\Phi_5$  (blue solid curve) and  $\chi_5^\perp$  (purple dashed curve) corresponding the eigenvalues  $E = -1$  and  $\lambda = -3/2$ , respectively, are shown. This system is governed by a non-symmetric double well potential, where a particle in the ground state is localized around the deeper well meanwhile a particle in the excited level is localized in the second well with a low probability to be found inside the deeper well.

## 5 Concluding remarks

We have constructed a recursive formula for the Wronskian of transformation functions, as they appear in the confluent SUSY algorithm. As byproducts, we obtained factorizations of the Wronskian and alternative representations of SUSY transformations. The main application of our results lies in the facilitation of establishing regularity conditions for potentials that were obtained through confluent SUSY transformations. While a complete analysis of such regularity conditions is beyond the scope of the present work, our results can prove useful when dealing with particular quantum systems. Besides the already well-known Pöschl-Teller model studied in section 4, our Wronskian representation (21) and the emerging regularity conditions are applicable in different contexts. As a recent interesting example for such an application, let us mention the work [12]. Here, a fourth-order confluent SUSY algorithm is applied to the free-particle system, generating Neumann-Wigner-type potentials that allow for the existence of so-called bound-states in the continuum. Since the underlying confluent SUSY transformation is of order four, the Wronskian of the transformation functions can be represented and analyzed using our recursive formula (21).

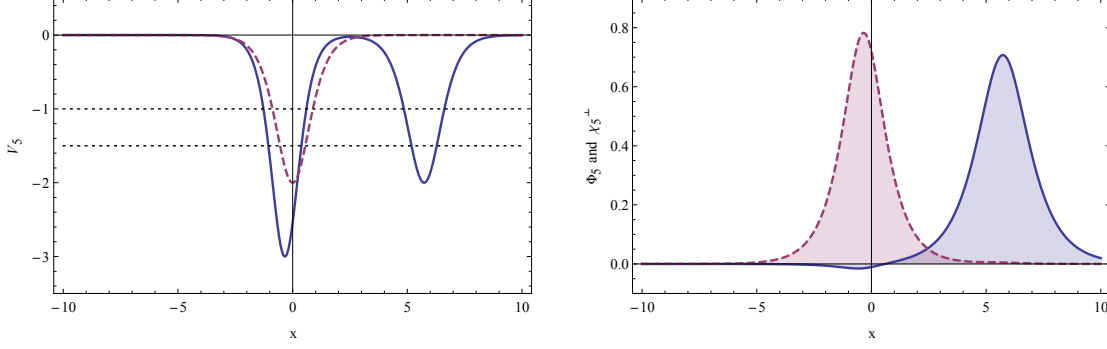


Figure 2: Left: Fifth-order SUSY partner of the hyperbolic Pöschl-Teller potential (blue solid curve). The Pöschl-Teller potential is plotted as reference (purple dashed curve). The parameters used are  $C_b = 0.01$ ,  $\kappa = \sqrt{3/2}$ . Right: functions  $\Phi_5$  (blue solid curve) and  $\chi_5^{1/2}$  (purple dashed curve) corresponding to the potential on the left.

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